Part II: Analysis and Recommendations: Mathematics Section of the Brazilian National Learning Standards
By Phil Daro – December, 2015
Analysis and Recommendations:
Mathematics Section of the Brazilian National Learning Standards\(^1\)

This analysis will summarize and formulate the most important recommendations for improving a good draft. Because the purpose of the analysis is to inform revision of the draft, it will point in the direction of what needs changing. This clarification for improvement should not be interpreted as heavy criticism of the draft. It is a good draft. It does need systematic attention toward making progressions of topics through the grades more coherent. Our recommendations are included below:

**Mathematical Expertise**

1. The Introductory text for Mathematics is eloquent. It will support aspirations but, like any text rich with meaning, it will suffer the degradations that come from an over-advised and under-resourced distribution of practitioners working under the pressures of time and difficult conditions. To increase the impact and practicality of the ideas in the introduction, an enumerated list should be distilled from the text and declared “Standards”. The Common Core in the U.S.A. with similar intent created eight “Standards of Mathematical Practice”. Each describes a particular expertise students should develop over and above learning content.

   We recommend Brazil do something similar, using its introductory text as an introduction to Standards of Practice. Without enumeration and formal designation as Standards, the management systems surrounding instruction cannot digest practices. Avoid mixing teaching practices with student practices. The Standards of Practice should address student expertise exclusively. Keep the number of expertise Practice standards to seven +/- 2. It is best if different versions can be constructed for elementary and secondary.

**Progression across grade levels**

2. Although many of the standards are well made, there is too often a lack of well designed progression across grade levels in particular domains where there are important dependencies of new knowledge and expertise on prior knowledge and expertise. This is most severe in elementary and lower secondary. Within a particular domain, for example ‘fractions’, the standards should form a sensible progression over the grades.

\(^1\) This analysis refers to the first draft of the mathematics section of the Brazilian National Learning Standards which was released in September of 2015. This document was commissioned by the Lemann Center at Stanford University by mathematics expert Phil Daro in December of 2015.
The importance of progressions goes beyond the obvious need to build adequate foundations of knowledge each year for successive years. Even in favorable circumstances, each classroom will have many students who function at below grade level stages of a progression for a given problem on a given day. Consequently, teachers have to work with a multi-grade stretch of a progression everyday. This is a normal condition in every nation. Having a well designed progression in the Standards can make this everyday job of the teacher more practical and successful.

I strongly recommend an organized and deliberate project to write progressions across grades for each of the high priority domains (see Recommendation 3, below for suggested high priority domains). The standards themselves should be derived from these progressions.

Example, Fractions:
Standards for fractions should be based on a progression. A fractions progression, for example, begins from concrete part/whole and sharing situations. But students should not be lead into a maze of complexities in the world of part/whole and sharing. Instead, they should develop concepts of fractions as an extension of prior understanding of number and measurement, as should be explicit in the standards. Visual models play a critical role with fractions and should be explicit in the standards, especially in grades 3-5. As efficiently as possible, visual models should progress toward deepening knowledge of the number line.

By defining unit fractions as numbers, the properties of numbers already learned can be extended to fractions. It is essential to make this explicit. We recommend doing this in a grade 3 standard. Unit fractions are numbers. The representation of quantities and numbers already in use should be extended to fractions in a systematic fashion. In particular, the number line is essential for understanding fractions as numbers. And fractions are essential for understanding the number line.

Yet the number line is difficult mathematics in its own right, so it needs its own development. This development begins with measurement of length and the sense of numbers that grows from measurement. Work with rulers and length in grades 1 and 2 are important building blocks for the number line. Adding and subtracting lengths, student drawn diagrams of operations with rulers should be explicitly part of grade 2 standards.

In grade 3, unit fractions can be defined concretely as numbers on the number line obtained from partitioning the length from 0 to 1 into equal parts. Comprehending this definition will take time and depends on prior work with concrete part/whole tasks, and equal sharing, partitioning tasks. The number \( \frac{1}{4} \) is the point exactly one partition from 0 when the length from 0 to 1 is partitioned into 4 equal parts. The number \( \frac{1}{4} \) behaves just like the numbers 1,2,3 do on the number line. Students can also learn that numbers have more than one name. \( \frac{1}{2} \) and \( \frac{2}{4} \) are two names for the
same number. 2/2, 3/3, and 4/4 are different names for 1. Restrict to simple fractions. That’s enough for grade 3.

In grade 4, students go further into equivalence, relying heavily on concrete and visual models. They learn how to generate equivalent fractions by multiplying numerator and denominator by the same number. They understand why this makes sense with visual models including the number line. They also learn that ¼ + 1/4 + ¼ = ¾. Any fraction can be written as the sum of unit fractions. Unit fractions can be counted just as students learn to count 10s (32 is 3 tens and 2 ones) or objects. 3 tens + 4 tens is 7 tens. 3 quarters + 4 quarters is 7 quarters (7/4). They also learn that ¾ = 3 x ¼. Any fraction can be written as the product of a whole number and a unit fractions.

In Grade 5, work with unit fractions and the number line flows together with work on equivalent fractions to form the basis for arithmetic with fractions with unlike denominators. Students learn to add and subtract fractions with unlike denominators by changing the fractions to equivalent fractions with like denominators. Like denominators means the same unit fractions. Adding and subtracting are a direct extension of arithmetic with whole numbers once we have the same unit fractions. Note: “least common denominator” is an unnecessary distraction from the more important idea of equivalence and is best omitted from the standards.

Also in grade 5, the area model used for whole numbers can should be extended for fractions. The unit squares which compose the area of rectangles with integer sides can be partitioned according to the definition of fractions. ¾ x 2/5 results in the 1 x 1 square partitioned on one side into 4 parts, and the other side into 5 parts. The resulting ¼ by 1/5 rectangles can be used to compose the product as a fraction of 1x1, 1 square unit. This extension of area model should be explicit to strengthen the progression.

In grade 6 and on, the value of a ratio a:b is the number a/b. This can only make sense to students if they have a firm grasp that a/b is a number.

**Priorities for Progressions**

3. In elementary grade standards, we recommend prioritizing the following domains for progression design:

   Number
   1. Base 10 decimal system and calculation with base 10 numbers
   2. Operations and algebraic reasoning (writing number expressions and equations for different types of problems)
   3. Fractions
   4. Measurement and magnitude
   4. Length measurement, numbers from measurement, operations (+,-,x,/) on measures using rulers and visual diagrams leading to number line
5. Length as a representation of quantities with non length units, such as elapsed time, leading to representation of rates with derived units in coordinate plane.

Geometry
6. Reasoning with properties of shapes
7. Area, composing and decomposing areas, support for operations with numbers (area model of multiplication, distributive property, etc)

Statistics
8. Data as a context for deepening expertise with number, table structures and visual representations of number situations and graphs.

Data and Statistics
4. Statistics is a wonderful and important domain for all students. Yet, its development over grades can be a challenge. Ideas like randomness, probability, independence and conditional require adequate cognitive readiness. There are many questions about developmental readiness for these ideas. We should not rush them into grade levels where many students will not be ready. Because time has to be budgeted in any case, work with data in the early grades should have modest ambitions vis a vis understanding statistics. Instead, data tasks should be used freely to extend and deepen understanding of number. Data situations can be naturally motivating for students. As students mature, the core ideas of statistics can be introduced and developed, but this may well come after grade 5. The progression for statistics should be developed with these considerations in mind.

Measurement
5. Aspects of measurement that support understanding of number, magnitude and relationships among magnitudes should be developed coherently. Of particular importance is the foundational role of length for representing number (and eventually variables) on number lines and in coordinate spaces. For these concepts and representational tools to be accessible to students, they need experiences with measurement of length as a representation of other quantities, for example elapsed time. This should be explicit in the standards.

An important progression extends from measurement to the study of rates in grades 6,7 and 8. This culminates in double number lines coordinated at 0 and coordinate graphs. Analysis of units and derived units should be explicit as part of making sense of real world situations involving relationships among quantities.

Functions
6. There should be more emphasis on functions in 8th and 9th grade in exchange for less emphasis on solving equations. A progression drawing on quantities from measurement to variables in proportional relationships to linear functions and their graphs can be well supported by concrete situations. Ratios should be explicated in
the standards as a building block for proportional relationships rather than as a special type of problem to learn how to solve by special methods.